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Systems of Linear Equations

Basic Definitions:

(i) Linear Equations and Solutions

A linear equation in unknowns $x_1, x_2, \dots., x_n$ is an equation that can be put in the standard form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \dots\dots\dots\dots\dots (1)$$

where $a_1, a_2, \dots, a_n$ are constants. The constant a_k is called the coefficient of x_k, and b is called the constant term of the equation.

A solution of the linear equation (1) is a list of values for the unknowns, say

$$x_1 = k_1, x_2 = k_2, \dots, x_n = k_n$$

such that the following statement is true:

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

In this case we say that $(k_1, k_2, \dots, k_n)$ satisfies the equation (1).

(ii) Systems of Linear Equations:

A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of m equations $L_1, L_2, \dots, L_m$ in n unknowns $x_1, x_2, \dots., x_n$ can be put in the standard form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where the a_{ij} and b_i are constant.

This system is called an $m \times n$ system. It is called square system if $m = n$, that is number of equations is equal to the number of unknowns. The number a_{ij} is called the coefficient of the unknown x_j in the equation L_i and the number b_i is called the constant of the equation L_i.

The system is called homogeneous if all the constant terms are zero, that is, if $b_1 = 0$, $b_2 = 0, \dots., b_m = 0$. Otherwise the system is said to be non-homogeneous.

A solution of a linear system is a tuple $(s_1, s_2, \ldots, s_n)$ of numbers that makes a each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$. The set of all solutions of a linear system is called the solution set of the system.

Note: The system of equation defined above can be written as $AX = B$, where A is given by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix},$$

Theorem 1.1. *Any system of linear equations has one of the following exclusive conclusions.*

(a) *No solution.*

(b) *Unique solution.*

(c) *Infinitely many solutions.*

A linear system is said to be **consistent** if it has at least one solution; and is said to be **inconsistent** if it has no solution.

Geometrical Interpretation:

The following three linear systems

$$(a)\begin{cases} 2x_1 & +x_2 & = & 3 \\ 2x_1 & -x_2 & = & 0 \\ x_1 & -2x_2 & = & 4 \end{cases} \quad (b)\begin{cases} 2x_1 & +x_2 & = & 3 \\ 2x_1 & -x_2 & = & 5 \\ x_1 & -2x_2 & = & 4 \end{cases} \quad (c)\begin{cases} 2x_1 & +x_2 & = & 3 \\ 4x_1 & +2x_2 & = & 6 \\ 6x_1 & +3x_2 & = & 9 \end{cases}$$

have no solution, a unique solution, and infinitely many solutions, respectively. See Figure 1.

Note: A linear equation of two variables represents a straight line in $\mathbb{R}^2$. A linear equation of three variables represents a plane in $\mathbb{R}^3$.In general, a linear equation of n variables represents a hyperplane in the n-dimensional Euclidean space $\mathbb{R}^n$.

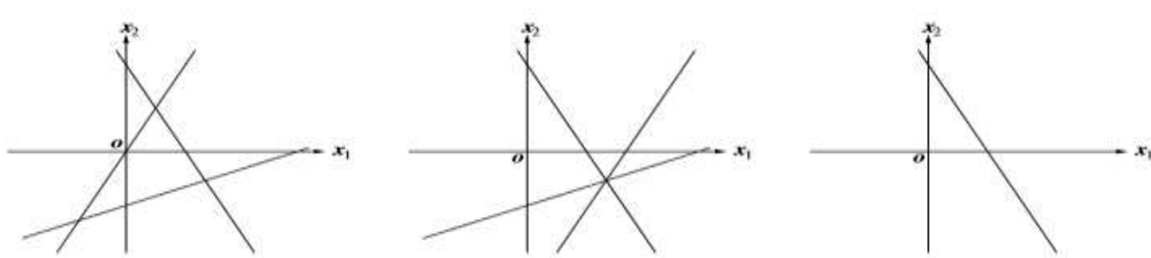

Figure 1: No solution, unique solution, and infinitely many solutions.

Augmented and Coefficient Matrices of a System:

Consider again a system of m equations in n unknowns. Such a system is associated with it the following two matrices:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

and

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The first matrix is called the augmented matrix of the system and is denoted by $[A{:}\,B]$. The second matrix is called the coefficient matrix of the system and is denoted by A.

Elementary Row Operations

There are three kinds of elementary row operations on matrices:

(a) Adding a multiple of one row to another row;

(b) Multiplying all entries of one row by a nonzero constant;

(c) Interchanging two rows.

Definition. Two linear systems in same variables are said to be equivalent if their solution sets are the same. A matrix A is said to be row equivalent to a matrix B , written A~B, if there is a sequence of elementary row operations that changes A to B.

Theorem. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set. In other words, elementary row operations do not change solution set.

Row Reduction and Echelon form:

A rectangular matrix is in echelon form (or row echelon form) if it has following three properties:

(i) All nonzero rows are above any rows of all zeros.

(ii) Each leading entry of a row is in a column to the right of the leading entry of row above it.

(iii) All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

(iv) The leading entry in each non-zero row is 1.

(v) Each leading 1 is the only non-zero entry in its column.

The following are two typical row echelon matrices.

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the circled stars $\bullet$ represent arbitrary nonzero numbers, and the stars $*$ represent arbitrary numbers, including zero. The following are two typical reduced row echelon matrices.

$$\begin{bmatrix} 1 & 0 & * & * & 0 & * & 0 & * & * \\ 0 & 1 & * & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition. If a matrix A is row equivalent to a row echelon matrix B, we say that A has the row echelon form B; if B is further a reduced row echelon matrix, then we say that A has the reduced row echelon form B.

Definition: A pivot position in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Theorem. Every matrix is row equivalent to one and only one reduced row echelon matrix. In other words, every matrix has a unique reduced row echelon form.

Existence and Uniqueness theorem:

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, that is, if and only if an echelon form of the augmented matrix has no row of the form [0…0 b], with b non-zero.

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable (A variable in a consistent linear system is called free if its corresponding column in the coefficient matrix is not a pivot column.)

Steps for solving a non-homogenous system:

(i) Write the augmented system of the system.

(ii) Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

(iii) Continue row reduction algorithm to obtain the reduced echelon form.

(iv) Write the system of equations corresponding to the matrix obtained in Step (iii).

(v) Rewrite each non-zero equation from Step (iv) so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Examples

1. Solve the following system of equations:

$$x_1 + 2x_2 - x_3 = 1$$

$$2x_1 + x_2 + 4x_3 = 2$$

$$3x_1 + 3x_2 + 4x_3 = 1$$

Solution. Perform the row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ \sim \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{array}\right] \begin{array}{c} (-1/3)R_2 \\ \sim \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ \sim \\ R_2 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system is equivalent to

$$\begin{cases} x_1 &= 7 \\ x_2 &= -4 \\ x_3 &= -2 \end{cases}$$

which means the system has a unique solution.

2. Solve the linear system:

$$x_1 - x_2 + x_3 - x_4 = 2$$

$$x_1 - x_2 + x_3 + x_4 = 0$$

$$4x_1 - 4x_2 + 4x_3 = 4$$

$$-2x_1 + 2x_2 - 2x_3 + x_4 = -3$$

Solution. Do the row operations:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 - 4R_1 \\ \sim \\ R_4 + 2R_1 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} (1/2)R_2 \\ R_3 - 2R_2 \\ \sim \\ R_4 + (1/2)R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \sim \end{array} \left[\begin{array}{cccc|c} (1) & [-1] & [1] & 0 & 1 \\ 0 & 0 & 0 & (1) & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system is equivalent to

$$\begin{cases} x_1 &= 1 + x_2 - x_3 \\ x_4 &= -1 \end{cases}$$

We see that the variables x_2, x_3 can take arbitrary numbers; they are called **free variables**. Let $x_2 = c_1$, $x_3 = c_2$, where $c_1, c_2 \in \mathbb{R}$. Then $x_1 = 1 + c_1 - c_2$, $x_4 = -1$. All solutions of the system are given by

$$\begin{cases} x_1 &= 1 + c_1 - c_2 \\ x_2 &= c_1 \\ x_3 &= c_2 \\ x_4 &= -1 \end{cases}$$

The general solutions may be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

Set $c_1 = c_2 = 0$, i.e., set $x_2 = x_3 = 0$, we have a **particular solution**

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

3. The linear system with the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right]$$

Solution: The given system has no solution because its augmented matrix has the row echelon form

$$\left[\begin{array}{ccc|c} (1) & 2 & -1 & 1 \\ 0 & (-3) & [7] & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The last row represents a contradictory equation 0 =−2.

4. Solve the linear system whose augmented matrix is

$$A = \left[\begin{array}{cccccc|c} 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 2 & 4 & -2 & 4 & -6 & -5 & -4 \end{array}\right]$$

Solution. Interchanging Row 1 and Row 2, we have

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 2 & 4 & -2 & 4 & -6 & -5 & -4 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \\ \sim \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 2 & -4 & -5 & -6 \end{array}\right] \begin{array}{c} R_2 \leftrightarrow R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -2 & 2 & -4 & -5 & -6 \end{array}\right] \begin{array}{c} R_4 + 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & -3 & -6 \end{array}\right] \begin{array}{c} R_4 + \frac{3}{2}R_3 \\ \sim \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_2 - R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{cccccc|c} (1) & [2] & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & (1) & [-1] & [2] & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & (1) & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the system is equivalent to

$$\begin{cases} x_1 = 1 - 2x_2 - x_4 + x_5 \\ x_3 = -2 + x_4 - 2x_5 \\ x_6 = 2 \end{cases}$$

The unknowns x_2, x_4 and x_5 are free variables.

Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$, where c_1, c_2, c_3 are arbitrary. The general solutions of the system are given by

$$\begin{cases} x_1 = 1 - 2c_1 - c_2 + c_3 \\ x_2 = c_1 \\ x_3 = -2 + c_2 - 2c_3 \\ x_4 = c_2 \\ x_5 = c_3 \\ x_6 = 2 \end{cases}$$

The general solution may be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Different forms of linear systems

A general system of linear equations is given by

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We introduce the column vectors:

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \ldots, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}; \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix};$$

and the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n].$$

Then the given system can be expressed by:

(a) The vector equation form: $x_1\boldsymbol{a_1} + x_2\boldsymbol{a_n} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{b}$.

(b) The matrix equation form: $Ax = b$.

(c) The augmented matrix form: $[\boldsymbol{a_1}\ \boldsymbol{a_1}\quad \boldsymbol{a_1} : \boldsymbol{b}]$.

Theorem. The system $Ax = b$ has a solution if and only if b is a linear combination of the column vectors of A.

Theorem. Let A be an m×n matrix. The following statements are equivalent.

(a) For each b in R^m, the system $Ax = b$ has a solution.

(b) The column vectors of A span R^m.

(c) The matrix A has a pivot position in every row.

Example: **The following linear system has no solution for some vectors b in R^3.**

$$\begin{aligned} 2x_2 + 2x_3 + 3x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 7x_4 &= b_2 \\ x_1 + x_2 + 2x_3 + 2x_4 &= b_3 \end{aligned}$$

The row echelon matrix of the coefficient matrix for the system is given by

$$\begin{bmatrix} 0 & 2 & 2 & 3 \\ 2 & 4 & 6 & 7 \\ 1 & 1 & 2 & 2 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 4 & 6 & 7 \\ 0 & 2 & 2 & 3 \end{bmatrix} \overset{R_2 - 2R_1}{\sim}$$
$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 2 & 2 & 3 \end{bmatrix} \overset{R_3 - R_2}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the following systems have no solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_3 + R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 2 & 2 & 3 & 1 \end{array}\right] \overset{R_2 + 2R_1}{\sim}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 2 & 4 & 6 & 7 & 0 \\ 0 & 2 & 2 & 3 & 1 \end{array}\right] \overset{R_3 \leftrightarrow R_1}{\sim} \left[\begin{array}{cccc|c} 0 & 2 & 2 & 3 & 1 \\ 2 & 4 & 6 & 7 & 0 \\ 1 & 1 & 2 & 2 & 0 \end{array}\right].$$

Thus the original system has no solution for $b_1 = 1$, $b_2 = b_3 = 0$.

Homogeneous system

A linear system is called homogeneous if it is in the form $Ax = 0$, where A is an m×n matrix and 0 is the zero vector in R^m. Note that $x = 0$ is always a solution for a homogeneous system, called the zero solution (or trivial solution); solutions other than the zero solution 0 are called nontrivial solutions.

Theorem. A homogeneous system $Ax = 0$ has a nontrivial solution if and only if the system has at least one free variable.

Note: If number of equations is equal to the number of unknowns, then the system has a non-trivial solution if $|A| = 0$.

Steps for solving homogeneous system:

(i) Reduce the matrix in its echelon form. If number of pivot entries is equal to the number of unknowns, then the system has only zero solution; Otherwise, there exists at least one non-trivial solution.

(ii) If there exists a non-trivial solution, then express each basic variable in terms of any free variables appearing in an equation.

(iii) Decompose the solution into a linear combination of vectors using free variables as parameter.

Examples

1. Find the solution set for the homogeneous linear system:

$$x_1 - x_2 - x_4 + 2x_5 = 0$$
$$-2x_1 + 2x_2 - x_3 - 4x_4 - 3x_5 = 0$$
$$x_1 - x_2 + x_3 + 3x_4 + x_5 = 0$$
$$-x_1 + x_2 + x_3 + x_4 - 3x_5 = 0$$

Solution. Do row operations to reduce the coefficient matrix to the reduced row echelon form:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ -2 & 2 & -1 & -4 & -3 \\ 1 & -1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & -3 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \\ \sim \\ R_4 + R_1 \end{matrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \begin{matrix} (-1)R_2 \\ R_3 + R_2 \\ \sim \\ R_4 + R_2 \end{matrix}$$
$$\begin{bmatrix} (1) & [-1] & 0 & 1 & 2 \\ 0 & 0 & (1) & [2] & [-1] \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the homogeneous system is equivalent to

$$\begin{cases} x_1 = x_2 - x_4 - 2x_5 \\ x_3 = -2x_4 + x_5 \end{cases}$$

The variables x_2, x_4, x_5 are **free** variables. Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 - 2c_3 \\ c_1 \\ -2c_2 + c_3 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Set $x_2 = 1, x_4 = 0, x_5 = 0$, we obtain the basic solution

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $x_2 = 0, x_4 = 1, x_5 = 0$, we obtain the basic solution

$$\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Set $x_2 = 0, x_4 = 0, x_5 = 1$, we obtain the basic solution

$$\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The general solution of the system is given by

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3, \quad c_1, c_2, c_3, \in \mathbb{R}.$$

Theorem. Let $Ax = 0$ be a homogeneous system. If u and v are solutions, then the addition and the scalar multiplication $u + v, cu$ are also solutions. Moreover, any linear combination of solutions for a homogeneous system is again a solution.

Theorem. Let $Ax = 0$ be a homogeneous linear system, where A is an m×n matrix with p pivot positions. Then system has $n - p$ free variables and $n - p$ basic solutions. The basic solutions can be obtained as follows: Setting one free variable equal to 1 and all other free variables equal to 0.

Linear Dependence and Independence:

Definition. Vectors $v_1, v_2, \dots, v_k$ in R^n are said to be linearly independent provided that, whenever

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

for some scalars $c_1, c_2, \dots, c_k$, then $c_1 = c_2 = \cdots = c_k = 0$. The vectors $v_1, v_2, \dots, v_k$ are called linearly dependent if there exist constants $c_1, c_2, \dots, c_k$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

Examples

1. **The vectors**

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^3$$

are linearly independent.

Solution. Consider the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system has the only zero solution $x_1 = x_2 = x_3 = 0$. Thus $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

2. **The vector**

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \text{ in } \mathbb{R}^3$$

are linearly independent.

Solution. Consider the linear system

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The system has one free variable. There is nonzero solution. Thus $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent.

Exercise: The vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

are linearly dependent.

3. **Any set of vectors containing the zero vector 0 is linearly dependent.**

Theorem: Let $v_1, v_2, \ldots, v_p$ be vectors in R^n. If $p > n$, then $v_1, v_2, \ldots, v_p$ are linearly dependent.

Proof: Let A = $[v_1, v_2, \ldots, v_p]$. Then A is an $n \times p$ matrix, and the equation $Ax = 0$ has n equations in p unknowns. Recall that for the matrix A the number of pivot positions plus the number of free variables is equal to p, and the number of pivot positions is at most n. Thus, if $p > n$, there must be some free variables. Hence $Ax = 0$ has nontrivial solutions. This means that the column vectors of A are linearly dependent.

Theorem. Let S =$\{v_1, v_2, \ldots, v_p\}$ be a set of vectors in R^n, (p ≥2). Then S is linearly dependent if and only if one of vectors in S is a linear combination of the other vectors. Moreover, if S is linearly dependent and $v_1 \neq 0$, then there is a vector v_p with $j \geq 2$ such that v_j is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{j-1}$.

Theorem. The column vectors of a matrix A are linearly independent if and only if the linear system

$$Ax = 0$$

has the only zero solution.

Proof. Let A = [$\boldsymbol{a_1}$, $\boldsymbol{a_2}$,..., $\boldsymbol{a_n}$]. Then the linear system $Ax = 0$ is the vector equation

$$c_1\boldsymbol{a_1} + c_2\boldsymbol{a_2} + \cdots + c_n\boldsymbol{a_n} = 0$$

Then $\boldsymbol{a_1}$, $\boldsymbol{a_2}$,..., $\boldsymbol{a_n}$ are linear independent is equivalent to that the system has only the zero solution.