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PAPER NAME- REAL ANALYSIS

TOPIC- REAL NUMBERS(Unit-I)

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Definition:

Algebraic Structure: A non-empty set with one or more composition (or operations) defined on it is called an "*algebraic structure"* or "*algebraic systems"*.

For Example: If A is a non-empty set and '*' is a operation defined on A then, (A, *) denotes an algebraic structure.

It is read as '' *A equipped with operation* * '' or '' *A together with the composition* * ". i.e. (A, *).

We know that R denotes the set of real numbers, then, (R, +, .) is an algebraic structure with two composition namely " Addition'' and ' Multiplication'.

The set of real numbers as an ordered field:

let R denote the set of real numbers and the two binary operations addition and multiplication be denoted by '+' and '.' respectively.

First we will give Field Axioms: The Algebraic structure (R, +, .) satisfy the following axioms:

I.) ***Field Axioms***

i) *The addition axioms*:

A_1) ∀ a, b ∈ R, a + b ∈ R (Closure law of Addition)

A_2) ∀ a, b ∈ R, a + b = b + a (Commutative law of Addition)

A_3) ∀ a, b, c ∈ R, a + (b + c) = (a + b) + c (Associative law of addition)

A_4) ∀ a ∈ R ∃ 0 ∈ R such that a + 0 = 0 + a = a (Existence of additive identity)

This real number '0' is called the additive identity of R.

A_5) ∀ a ∈ R ∃ b ∈ R such that a + b = b + a = 0 (But a + b = 0 = b + a if b = - a i. e., additive inverse of a real number 'a' is its negative)

Thus, a +(- a) = 0 = - a + a

ii.) *<u>The Multiplication Axioms</u>*:

M_1) ∀ a, b ∈ R, a.b ∈ R (closure law of Multiplication)

M_2) ∀ a, b ∈ R, a.b = b.a (commutative law of Multiplication)

M_3) ∀ a, b, c ∈ R, a.(b.c) = (a.b).c (Associative law of Multiplication)

M_4) ∀ a ∈ R, ∃ 1 ∈ R such that a.1 = 1.a = a (here, the real number '1' is called the multiplication identity of R)

M_5) ∀ a ∈ R, a ≠ 0, ∃ b ∈ R such that a.b = b.a = 1.

The real number b is called the multiplicative inverse of a and is denoted by a^{-1} or $\frac{1}{a}$. which is reciprocal of a.

iii.) *<u>Distributivity</u>*:

Multiplication distributes over addition of real numbers.

D.) ∀ a, b, c ∈ R,

a.(b + c) = a.b + a.c

***NOTE* : A non-empty set containing atleast two elements and with two binary operations satisfying all the eleven Axioms A_1 to A_5, M_1 to M_5 and D is called a field.**

II.) Order Axioms:

O_1. ∀ a, b ∈ R, exactly one of the following holds.

i.) a > b ii.) a = b or iii.) a < b

This law is known as Trichotony law.

O_2. ∀ a, b, c ∈ R a > b and b > c → a > c

This is transitivity law.

O_3 . ∀ a, b, c ∈ R , a > b → a+c > b+c ∀ c ∈ R

This is known as Monotone law of addition.

O_4. ∀ a, b ∈ R, a > b → ac > bc if c > 0

This is known as Monotone law of multiplication.

Ordered Field:

If a field satisfies all the order axioms O_1 to O_4, then it is called an ordered filed.

Thus, (R, +, .) is an ordered field.

Similarly, we can say that (Q, +, .) is an ordered field.

***NOTE*: Multiplicative inverse exists only for a Non-zero real number. (0 has no multiplicative inverse).**

Some Important Theorems:

Theorem 1: Prove that in the set of real numbers:

a.) Addition identity is unique.
b.) Multiplicative identity is unique.
c.) Additive inverse is unique.
d.) Multiplicative inverse is unique.

Proof a.)

We know that 0 is the additive identity in R. Such that

a + 0 = 0 + a = a, ∀ a ∈ R.

If possible, let us suppose that 0 and $0'$ are two additive identities in R. Then, if 0 ∈ R $0'$ is additive identity then,

$0 + 0' = 0$ ……………(1)

If $0' \in R$ and 0 is the additive identity, then

$0' + 0 = 0'$ ……………….(2)

But, we know

$0 + 0' = 0' + 0$

$\rightarrow 0 = 0'$ (by (1) and (2)

Thus, the additive inverse is unique.

Proof c.)

To prove: Addition inverse in R is unique.

We know that b is the additive inverse of a

If a + b = b + a = 0, where 0 is the additive identity of R

Let a ∈ R and it has two additive inverse b and c

Then, by definition of additive inverse,

a + b = 0 and a + c = 0

To prove: b = c

Now,

b = b + 0 [since, 0 is additive identity]

= b + (a + c) [a + c = 0]

= (b + a) + c [by associativity property of R]

= (a + b) + c [by commutativity property of R]

= 0 + c

= c [since, 0 is additive identity 0 + c = c + 0 = c].

→ b = c

Theorem 2: Similarly, using the properties of Addition and Multiplication of the system of real numbers, prove that

i. If a + c = a + b then, b = c (Cancellation laws)
ii. If a.b = a.c then, b = c (a$\neq$ 0)(Cancellation laws)
iii. –(-a) = a

Note: Give reasons at each step.

Theorem 3: $\forall$ x, y $\in$ R, prove that

i. xy > 0 → either (x > 0 and y > 0) or (x < 0 and y < 0)
ii. xy < 0 → either (x < 0 and y > 0) or (x > 0 and y <0)

i.) Since, x, y ∈ R

There are three possibilities with them,

a. Either x = 0, x> 0 or x < 0
b. Either y = 0 or y > 0 or y < 0,

Here, since xy > 0. Then, x $\neq$ 0, y $\neq$ 0.

Thus, we are left with only two possibilities.

So, let xy > 0. Also suppose that x > 0 and x^{-1} exists as x $\neq$ 0.

Since, x > 0 → x^{-1} > 0.

Thus, xy > 0

→ x^{-1}(xy) > x^{-1}.0 [We know that a, b ∈ R and c > 0, ac > bc]

→ 1.y > 0 [1 is the multiplicative inverse thus, x $\neq$ 0 → xx^{-1} = x^{-1}x = 1][1.y = y.1 = y ∀ y ∈ R]

xy > 0 → x > 0 and y > 0.

Secondly, let xy > 0 and x < 0.

Then, x^{-1} exists and x^{-1}< 0

xy > 0 → x^{-1}(xy) < x^{-1}.0

→ (x^{-1}.x).y < 0 [by associativity of multiplication in R]

→ 1.y < 0 [Since, 1 is the multiplicative identity, x^{-1}.x = 1 ∀ x ∈ R]

→ y < 0 [1.y = y.1 = y ∀ y ∈ R]

Hence, xy > 0 → either (x > 0 and y > 0) or (x < 0 and y< 0)

Conversely, suppose that x > 0, y > 0, then, to prove xy > 0.

x > 0 → xy > o.y as y > 0 [∀ a, b, c ∈ R, a > b → ac > bc if c > 0]

→ xy > 0.

Again, let x < 0 , y < 0

→ (-x) > 0 & (-y) > 0

By the first one, (-x)(-y) > 0. Hence, the result.

Theorem 4:

Define a rational number and prove that $\sqrt{2}$ is not a rational number.

Or,

Prove that $\sqrt{2}$ is irrational.

Or,

Prove that there exists no rational number whose square is 2.

Proof: A number of the form $\frac{p}{q}$, where p and q are integers having no common factor other than 1 and q ≠ 0 is called a rational number.

For example;

$\frac{3}{2}, \frac{-5}{3}, \frac{7}{8}, \frac{6}{1}$ etc. all are rational numbers. The set of rational numbers is denoted by Q.

If possible, let us suppose that $\sqrt{2}$ is a rational number.

Then, $\sqrt{2} = \frac{p}{q}$ .... (1)

Where p and q are integers and q $\neq$ 0, and prime to each other.

(i.e., p and q have no common factor other than 1)

Squaring (1), we get

$2 = \frac{p^2}{q^2}$

$\rightarrow p^2 = 2q^2$ ....(2)

$\rightarrow p^2$ is even as it is a multiple of 2. Hence, p is also even as we know that square of an even number is even. Let p = 2m, where m is an integer.

From (2),

$4m^2 = 2q^2$

$\rightarrow q^2 = 2m^2$

$\rightarrow q^2$ is even and hence q is also even.

Thus, from above we see that p and q both are even integers and they have a common factor 2 which contradicts our supposition that p and q have no common factor. Hence, $\sqrt{2}$ cannot be expressed in the form of $\frac{p}{q}$, therefore, $\sqrt{2}$ is not a rational number.

NOTE: If n is prime, use method of theorem 5 and if n is composite then use method of theorem 6 for proof.

Theorem 5:

Show that for any positive prime p different from 1, $\sqrt{p}$ is an irrational number.

Proof: If possible, let us suppose that $\sqrt{p}$ is a rational number,

Let $\sqrt{p} = \frac{a}{b}$, where a , b are +ve integers having no common factor except 1 and b $\neq$ 0.

Then, squaring we get

P = $\frac{a^2}{b^2}$

$\rightarrow b^2 p = a^2$

$\rightarrow\ a^2 = pb^2$

$\rightarrow$ The prime p divides a^2.

Since, p is prime p must divide a.

Hence, there exists a positive integer m such that a = pm

But, then,

p = $\frac{a^2}{b^2}$ $\rightarrow$ p = $\frac{p^2 m^2}{b^2}$

$\rightarrow\ b^2 = pm^2$

$\rightarrow$ p divides b^2

Since, p is prime, p must divide b.

Hence, there exists a positive integer n such that b = np

Thus, from above we see that a = pm and b = np. i.e. a and b both have a common factor p different from 1 which is against our supposition.

Therefore, $\sqrt{p}$ is not a rational number. Hence, $\sqrt{p}$ is an irrational number.

NOTE: We can use similar process for proof is p is a prime other than 1 i.e. $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$, $\sqrt{13}$ …………………so on.

This statement can also be written as; the equation x^2= 2 has no solution in the Rational number system.

Theorem 6:

Prove that $\sqrt{8}$ is not a rational number. Or, there exists no rational whose square is 8.

Proof:

We know that $\sqrt{4} < \sqrt{8} < \sqrt{9}$

→ $2 < \sqrt{8} < 3$

If possible, let us suppose that $\sqrt{8}$ is a rational number.

Let $\sqrt{8} = \frac{p}{q}$ where p and q are positive integers

With no common factor other than 1 and $q \neq 0$.

Now,

$2 < \sqrt{8} < 3$

→ $2 < \frac{p}{q} < 3$

→ $2q < p < 3q$ [$q > 0$]

→ $0 < p - 2q < q$

→ p – 2q is a positive integer less than q.

→ p – 2q and q are co-prime.

Also, p and q are co-prime.

Therefore,

$\sqrt{8}$ (p – 2q) = $\frac{p}{q}$(p – 2q) is not an integer.

But, $\sqrt{8}$ (p – 2q) = $\frac{p}{q}$(p – 2q) = $\frac{p^2}{q}$ - 2p

= $\sqrt{8}$.p – 2p

= 8.q – 2p [Since, $\frac{p}{q}$=$\sqrt{8}$, $\frac{p^2}{q^2}$ = 8.]

= an integer.

Thus, we arrive at a contradiction. Hence, $\sqrt{8}$ is not a rational number.

Greatest lower Bound and Least Upper Bound

Definition1: Least Upper Bound (lub) or Supremum

Let S be a set which is bounded above. Then, there exist an infinite number of upper bounds of S as every number greater than an upper bound is also an upper bound for S.

If the set of all upper bounds of set S has the least or smallest member say M, Then M is called least upper bound (l.u.b) or supremum of S.

The Supremum of S has the following properties

i.) $x \leq M \; \forall \, x \in S$ i.e., M is an upper bound of S.

ii.) Given$\in > 0$, however small, ∃ y ∈ S such that y > M-∈.

Definition2:

Greatest lower bound (g.l.b) or Infimum.

Let S be a set which is bounded below then, S admits an infinite number of lower bounds as every number less than any lower bound of S is also a lower bound of S.

If the set of all lower bounds has the greatest member, say m, then, ‘m’ is called the greatest lower bound(g.l.b) or Infimum of S.

The Infimum ‘m’ of S has the following properties;

i.) $x \geq m$ i.e. m is a lower bound of S.

ii.) Given $\in > 0$, however small ∃ y ∈ S, such that y < m + ∈

THE ORDER COMPLETENESS AXIOMS:

The least upper bound axiom: Every non-empty subset of R which is bounded above has the least upper bound or Supremum.

Equivalently,

The greatest lower bound axiom: Every non-empty subset of R which is bounded below has the greatest lower bound or Infimum.

l.u.b Axiom ↔ g.l.b Axiom.

Complete Ordered Field:

An ordered field which satisfies the completeness axiom is called a complete ordered field.

NOTE:

We have already shown R is an ordered field and in addition with the completeness axiom, it is a complete ordered field.

→ R is a complete ordered field.

Question:

State all the properties that make R a complete ordered field.

<u>Theorem on g.l.b and l.u.b</u>

Theorem 1:

Every non-empty subset of real numbers which is bounded above has the least upper bound.

Proof:

Let S be a non-empty set of real numbers which is bounded above. Let K be an upper bound of S. Let us consider a set defined as follows:

T = $\{-x : x \in S\}$

Since, S is non-empty set of real numbers, so is T. Since, K is an upper bound of S

$x \leq k \;\forall\; x \in S$.

→ $-x \geq -k \;\forall\; x \in S$

→ $-k \leq -x \;\forall\; x \in S$

→ $-k \leq y \;\forall\; y \in T$ Since, $y = -x \in T$.

This implies that T is bounded below and –k is a lower bound of T.

Since, T is a non-empty set of real numbers which is bounded below, thus by completeness axiom of real numbers, the set T has the greatest lower bound.

Let g.l.b T = l and u = - l

Since, l is the g.l.b of Tby definition

$l \leq y \quad \forall\ y \in T$

→ - $l \geq$ -y $\forall$ y ∈ T

Thus , u ≥ x $\forall$ x ∈ S [If y ∈ T ,then ,-y ∈ S]

→ $x \leq u \quad \forall\ x \in S$

→ u is an upper bound of S

Next to prove that u is the least upper bound of S.

For this it is sufficient to prove that there is no upper bound less than u of S.

OR

Any other upper bound of S other than u is always greater than u.

Let u'be another upper bound of S, then

$x \leq u' \quad \forall\ x \in S$

→ $-\ x \geq -u'\ \forall\ x \in S$

→ $y \geq -u'\ \forall\ y \in T$ [$x \in S$, $-x =\ y \in T$]

→ $y \geq -u'\ \forall\ y \in T$

→ $-u'$ is a lower bound of T.

Since, l is the g.l.b of T, by definition

$-u' \leq l$

→ $u' \geq -l$

$\rightarrow u' \geq u$, as $u = -l$

$\rightarrow u \leq u'$.

Since, u' was as arbitrary upper bound of S, thus every upper bound of S, other than u is greater than u

$\rightarrow u$ = l.u.b S.

Hence, l has the least upper bound.

Archimedean property of real numbers:

Statement: *If x is a positive real number and y is any real number, then there exists a positive integer n such that n $x > y$ i.e., ∀ x, y ∈ R ∃ n ∈ N such that n x > y.*

Another form

If x ∈ R, then exists n_x ∈ N such that x < n_x i.e., we can also write x < n.

OR

If x, y ∈ R and x > 0, y > 0, then there exists a natural number n such that ny > x or nx > y.

Proof:

Since, $x > 0$ and $y \in R$, i.e. y is any real number, by trichotomy law of real numbers, we have *three possibilities.*

$y > 0$ or $y = 0$ or $y < 0$.

We have to show that for all the cases $nx > y$ where n ∈ N.

Case 1.) If $y > 0$

If possible let us suppose that the theorem is false.

i.e $nx \leq y$ for every $n \in N$ and $x > 0$.

Then, the set **A =** $\{nx;\ n \in N\}$ is bounded above by y.

Thus by Order completeness property of R , **A** possesses least upper bound.

Let l.u.b **A** =u

Since, least upper bound of **A** is u,

$$nx \leq u \quad \text{for every } n \in N$$

$\rightarrow (n+1)x \leq u$ for every $n \in N$ [$n \in N$, $n+1 \in N$]

$\rightarrow nx \leq u - x$ for every $n \in N$

$\rightarrow nx \leq u - x < u$ for every $n \in N$

Thus, $(u - x)$ is a number which is strictly less than u and an upper bound of A i.e. we got an upper bound of A which is less than its l.u.b which contradicts the definition of least upper bound.

Thus, our supposition that $nx \leq y$, $\forall\, n \in N$ is wrong. Hence, $\exists$ an element $n \in N$ such that $nx > y$.

Case 2.) If $y \leq 0$, since $x > 0, n > 0$

$\rightarrow n\,x > 0$ by order property of R

$\rightarrow n\,x > 0 \geq y$

$\rightarrow nx > y$

Hence, the proof.

Densoness of R

I. Between any two distinct real numbers, there is always a rational number and therefore infinitely many rational numbers.

Proof:

Let us consider two distinct real numbers x & y.

Thus, $x \neq y$, hence two possibilities either $x < y$ or $x > y$

Let $x < y$ then, $y - x > 0$. Then, by Archimedean property of real numbers, $\exists\, n \in N$ such that

$n(y-x) > 1$, where $1 \in$ R i.e., $ny - nx > 1$

$\rightarrow ny > 1 + nx$ ....(i)

$also$, $\exists$ an unique integer m such that

$m - 1 \leq nx < m$ ....(ii)

$then$, from (i)

$ny > nx + 1 \geq m > nx$

$\rightarrow nx < m < ny$

$\rightarrow x < \frac{m}{n} < y$

$\frac{m}{n}$ = r, is a rational number between$x$ and y.

$thus,\ x < r < y$

Repeating the same arguments with $x < r$ and$r < y$, we get rational numbers r_1 and r_2 such that

$x < r_1 < r < r_2 < y$

Continuing, this process infinitely, we get infinitely many rational between$x$ and y.

Similarly, we can prove this for $x > y$ i.e., $y < x$.

Hence, there lie infinitely many rationals between two real numbers.

II. Between any two distinct real numbers, there is always an irrational number and therefore infinitely many irrational numbers.

Proof:

Let us consider two distinct real numbers x and y with $x < y$ and $x \neq y$.

$x < y \rightarrow y - x > 0$

By Archiemedean property of real numbers, $\exists\, n \in N$ such that

$\rightarrow n(y - x) > \sqrt{2}$ as $\sqrt{2} \in R$

$\rightarrow y - x > \frac{\sqrt{2}}{n}$

$therefore, y > x + \frac{\sqrt{2}}{n} > x + \frac{\sqrt{2}}{2n} > x.$

$since, \left(x + \frac{\sqrt{2}}{n}\right) - \left(x + \frac{\sqrt{2}}{2n}\right) = \frac{\sqrt{2}}{2n}$ is an irrational number.

$Atleast$ One of $\left(x + \frac{\sqrt{2}}{n}\right)$ and $(x + \frac{\sqrt{2}}{2n})$ is irrational. Let us denote it by r.

Then, $x < r < y$

Therefore, there is an irrational number between$x$ and y.

Repeating the same argument for $and\ r, \&\ r\ and\ y$, we get irrational numbers r_1 and r_2 such that

$x < r_1 < r < r_2 < y$

Continuing like this, we get infinitely many irrational numbers between two real numbers. Therefore, there are infinitely many irrational numbers between real numbers.

FUNDAMENTAL THEOREM OF ANALYSIS/ CLASSICAL ANALYSIS

If L and U are any two sets of real numbers such that

i) Neither L nor U is empty ;
ii) Every element of L is less than every element of U;
iii) Given any positive real number ε ,elements u in U and l in L are available such that u-l $< \varepsilon$,

then ,there exists a real number a such that lub L = glb U = a

Properties of Infimum and Supremum :

I)If A and B are bounded subsets of R , then prove that the set A+B ={ x+y: x ∈ A and y ∈ B } is also bounded and

(i) Sup.(A+ B) =Sup.A +Sup.B

(ii) Inf.(A+B) = Inf.A +Inf.B

II) For a real number , let λA ={ λx: x ∈ A }.Prove that if A is bounded subset of R, then λA is bounded and

i). Sup. λA = λ Sup.A if $\lambda > 0$

ii). Sup. λ A = λ Inf.A if $\lambda < 0$

iii). Inf. λ A = λ Sup.A if $\lambda < 0$

iv). Inf. λ A = λ Inf.A if $\lambda > 0$