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Numerical Solution of Ordinary Differential Equations

Introduction

- Many problems in science and engineering can be reduced to the problems of solving differential equations satisfying certain given conditions.
- The analytical methods of solving differential equations are applicable only to limited class of equations.
- Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods.
- These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

The general form of first order ODE with an initial condition is given by:

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

In this chapter, we will discuss following three numerical methods for finding an approximate solution of the above equation at some point:

(1) Euler's method
(2) Modified Euler's method
(3) Runge Kutta second and fourth order methods.

Euler's method:

Euler's method is useful for finding y(x) at some point x (for the differential equation (1)) or a tabulated set of values.

Suppose we wish to solve equation (1) for the values of y at $x = x_r = x_0 + rh, r = 1, 2, \ldots.$

Now integrating equation (1) within the limits x_0 and x_1, we obtain

$$\int_{x_0}^{x_1} y dx = \int_{x_0}^{x_1} f(x, y) dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Assuming that $f(x, y) = f(x_0, x_1)$ in $x_0 \leq x \leq x_1,$ this gives us

$$y(x_1) - y(x_0) = f(x_0, y_0) \int_{x_0}^{x_1} dx$$

$$\boxed{y_1 = y_0 + hf(x_0, y_0)}$$

Similarly, for the range $x_1 \leq x \leq x_2$, we obtain

$$\boxed{y_2 = y_1 + hf(x_1, y_1)}$$

Proceeding in this way, we obtain the general formula:

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

The process is slow and to obtain reasonable accuracy with Euler's method, we need to take a smaller value of *h*. Because of this restriction on h, the method is unsuitable for process for practical use and a modification of it, known as Modified Euler's method, which gives more accurate result, will be discussed later.

Examples

1. **Consider the differential equation**

$$\frac{dy}{dx} = -y, \qquad \text{with an initial condition } y(0) = 1$$

Find $y(0.01), y(0.02), y(0.03), y(0.04)$.

Solution: In this case, $f(x, y) = -y, x_0 = 0, y_0 = 1.$

Euler's formula is given by:

$$y_n = y_{n-1} + hf(x_{n-1}, x_{n-1})$$

$y(0.01) = y_0 + hf(x_0, y_0) = 1 + 0.01(-1) = 0.99$

$y(0.02) = y_1 + hf(x_1, y_1) = 0.99 + 0.01(-0.99) = 0.9801$

$y(0.03) = y_2 + hf(x_2, y_2) = 0.9801 + 0.01(-0.9801) = 0.9703$

$y(0.04) = y_3 + hf(x_3, y_3) = 0.9703 + 0.01(-0.9703) = 0.9606$

2. **Consider the initial value problem**

$$\frac{dy}{dx} = y - x, y(0) = \frac{1}{2}$$

Use Euler's method with h=0.1 to obtain y(0.5).

Solution: Given the initial value problem is of the form

$$y' = f(x, y), \quad y(0) = y_0$$

where $f(x, y) = y - x, \ x_0 = 0, y_0 = 1/2.$

Let h=0.1. So by Euler's formula,

$$y_1 = y(0.1) = y_0 + hf(x_0, y_0) = \frac{1}{2} + 0.1 \times f\left(0, \frac{1}{2}\right) = 0.55$$

$$y_2 = y(0.2) = y_1 + hf(x_1, y_1) = 0.55 + 0.1 \times f(0.1, 0.55)$$
$$= 0.55 + 0.1(0.55 - 0.1)$$
$$= 0.595$$

$$y_3 = y(0.3) = y_2 + hf(x_2, y_2) = 0.595 + 0.1 \times f(0.2, 0.595)$$
$$= 0.595 + 0.1(0.595 - 0.2)$$
$$= 0.6345$$

$$y_4 = y(0.4) = y_3 + hf(x_3, y_3) = 0.6345 + 0.1 \times f(0.3, 0.6345)$$
$$= 0.6345 + 0.1(0.6345 - 0.3)$$
$$= 0.66795$$

$$y_5 = y(0.5) = y_4 + hf(x_4, y_4) = 0.66795 + 0.1 \times f(0.4, 0.66795)$$

$$= 0.66795 + 0.1(0.66795 - 0.4)$$

$$= 0.694745$$

Therefore, $y(0.5) = 0.69474$

Modified Euler's method:

Instead of approximating *f(x,y)* by *f(x₀,y₀)* we now approximate the integral given in (2) by means of trapezoidal rule to obtain

$$y_1 = y_0 + h\left\{\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right\}$$

We thus obtain the iteration formula

$$y_1^{(n)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{n-1})]$$

n=0,1, 2…., where $y_1^{(n)}$ is the nth approximate of y_1. The Euler's formula will be used to obtain the initial iterate $y_1^{(0)}$ i.e.,

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

Examples

1. **Determine the value of y at x=0.1, given that $y' = x^2 + y, y(0) = 1$.**

Solution: Let h=0.05. It is given that $f(x, y) = x^2 + y, x_0 = 0, y_0 = 1$.

So $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05f(0,1) = 1.05$

Further $x_1 = x_0 + h = 0.05$ and $f\left(x_1, y_1^{(0)}\right) = 1.0525$. So next approximation $y_1^{(1)}$ for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right\} = 1 + \frac{0.05}{2}\{f(0, 1) + f(0.05, 1.05)\}$$

$$= 1.051312$$

Repeating the procedure, we obtain

$$y_1^{(2)} = 1.0513453$$

Hence, we take $y_1 = y(0.05) = 1.0513$(correct upto 4 decimal places)

Next, we take $x_1 = 0.05,\ y_1 = 1.0513$ and $h = 0.05$, we obtain

$$y_2^{(0)} = y^{(0)}(x+2h) = y^{(0)}(0.1) = y_1 + hf(x_1, y_1) = 1.0513 + 0.05f(0.05, 1.0513)$$
$$= 1.10399$$

Now,

$$y_2^{(1)} = y_1 + \frac{h}{2}\left\{f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right\}$$
$$= 1.0513 + \frac{0.05}{2}\{f(0.05, 1.0513) + f(0.1, 1.10399)\}$$
$$= 1.105495$$

Similarly, $y_2^{(2)} = 1.10553238$

Therefore, $y(0.1) = 1.1055$(correct upto 4 decimal places).

2. **Apply the modified Euler's method with h=0.1 to determine an approximation to the solution to the initial value problem at $x = 0.1$.**

$$\boldsymbol{y' = y - x, y(0) = \frac{1}{2}}$$

Solution: Let h=0.1. It is given that $f(x, y) = y - x, x_0 = 0, y_0 = 1/2$.

Initial approximation for $y(0.1)$ is obtained by the Euler's formula:

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = \frac{1}{2} + 0.1f\left(0, \frac{1}{2}\right) = 0.55$$

Further $x_1 = x_0 + h = 0.1$. So, first approximation $y_1^{(1)}$ for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right\} = \frac{1}{2} + \frac{0.1}{2}\left\{f\left(0, \frac{1}{2}\right) + f(0.1, 0.55)\right\}$$
$$= 0.5475$$

Second iterative value $y_1^{(2)}$ for $y(0.1)$ is

$$y_1^{(2)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right\} = \frac{1}{2} + \frac{0.1}{2}\left\{f\left(0, \frac{1}{2}\right) + f(0.1, 0.5475)\right\}$$
$$= 0.547375$$

Third iterative value $y_1^{(3)}$ for $y(0.1)$ is

$$y_1^{(3)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right\} = \frac{1}{2} + \frac{0.1}{2}\left\{f\left(0, \frac{1}{2}\right) + f(0.1, 0.547375)\right\}$$
$$= 0.54736875$$

Since $y_1^{(2)}$ and $y_1^{(3)}$ are same upto four decimal places (their values are 0.5474 correct upto decimal places), so we obtain that

$$y_1 = y(0.1) = 0.5474$$

Runge Kutta Method:

Given $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$.

(i) Runge Kutta second order method:

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

(ii) Runge Kutta fourth order method:

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Examples

1. **Given $\frac{dy}{dx} = y - x, y(0) = 2$. Find y (0.1), y(0.2) correct to four decimal places.**

Solution: Given $f(x,y) = y - x$, $x_0 = 0, y_0 = 2$.

(i) Runge Kutta second order method:

Let h=0.1. Then

$k_1 = hf(x_0, y_0) = 0.1(2 - 0) = 0.2$

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1f(0.1, 2 + 0.2) = 0.1(2.2 - 0.1) = 0.21$

And therefore,

$$y_1 = y(0.1) = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050$$

Next, to determine $y_2 = y(0.2), x_1 = 0.1, y_1 = 2.2050$

Hence

$k_1 = hf(x_1, y_1) = 0.1f(0.1, 2.2050) = 0.1(2.2050 - 0.1) = 0.2105$

$k_2 = hf(x_1 + h, y_1 + k_1) = 0.1f(0.2, 2.2050 + 0.2105) = 0.22155$

It follows that

$$y_2 = y(0.2) = y_1 + \frac{1}{2}(k_1 + k_2) = 2.2050 + \frac{1}{2}(0.2105 + 0.22155) = 2.421025$$

So, y (0.1)= 2.2050 and $y(0.2) = 2.421025$

(ii) Runge Kutta fourth order method:

To determine $y_1 = y(0.1), x_0 = 0, y_1 = 2$ and $h = 0.1$. We then obtain

$$k_1 = hf(x_0, y_0) = 0.1f(0,2) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 2.01) = 0.205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 2.1025) = 0.20525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.210525$$

Hence,

$$y_1 = y(0.1) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.20517$$

Proceeding similarly, we obtain $y(0.2) = 2.4214$

2. Given $\frac{dy}{dx} = 1 + y^2, y(0) = 0$. Find y (0.2), y(0.4) and y(0.6).

Solution: Given $f(x, y) = 1 + y^2$, $x_0 = 0, y_0 = 0$.

Let $h = 0.2$. So,

$k_1 = hf(x_0, y_0) = 0.2f(0,0) = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.1) = 0.202$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.101) = 0.20204$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.20204) = 0.20816$

which implies that

$$y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.202707$$

Next, to compute $y(0.4)$, $x_1 = 0.2, y_1 = 0.202707$ and h=0.2. Then

$k_1 = hf(x_1, y_1) = 0.2f(0.2,0.202707) = 0.208218$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 0.306816) = 0.218827$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 0.3121305) = 0.219484$$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.235650$

which implies that

$$y_1 = y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.422788667$$

Finally, to obtain $y(0.6)$, take $x_2 = 0.4, y_1 = 0.4228$ and h=0.2 and proceeding as above, we get $y(0.6) = 0.6841$.