Differential Equations & Mathematical Models

(e-content for UG)

Semester - II (Paper Code MATH CC204)

Mr. Ravi Kumar
Assistant Professor
Department of Mathematics
Patna Women’s College, Patna University.
Email.id: Kumar.ravi.iitg@gmail.com

Contents:
Unit : 1 of (MATHCC 204)

- Introduction
- Differential equation
- General and Particular Solution
- Singular Solution
- Mathematical models
- Linear equation and Bernoulli equations
- Exact differential equation and integrating factors and transformations.
- Exercise

objective

- To study the first order differential equations and solve differential equation by classical methods.
- The Focus is on those mathematical techniques that are applicable to models involving differential equations.
- The real world situation to be investigated. Think about all the physical, chemical, biological, social, economic laws that may be relevant to the situation.
- Understand the concepts of modeling with compartments and the balance law.
- Solution of models using classical method of solution as well as solution using mathematical software.

Introduction

- The Laws of the universe are written in language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by the equations that relate changing quantities.
- Because the derivative $\frac{dx}{dt} = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variables t, it is natural that equation involving derivatives are frequently used to describe the changing universe.

Differential equation

- Differential equations have wide applications in various science and engineering disciplines. In general modeling variations of a physical quantity such as displacement, velocity, pressure, temperature, stress, strain or concentration of a pollutant with the change of time t or location such as the coordinate (x, y, z) or both would require differential equation. Models of systems have become part of our everyday lives. Mathematical modeling essentially consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of the real world.

- Differential equations are important tools in dealing with problems related to applied mathematics and allied fields.
- Differential equation involves a functions and its derivative. It is classified into two parts
- ordinary differential equation abbreviated as O.D.E and partial differential equation abbreviated as P.D.E.
- While dealing with differential equations the order and degree of differential equation are important. The differential equations may be linear and nonlinear

Differential Equations:

Definition:

- A differential equation is an equation containing an unknown functions and its derivatives.

Examples:

1. $$\frac{dy}{dx} = 2x + 3$$

2. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

y is dependent variable and x is independent variable.

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Ordinary Differential equation:

- A differential equation is said to be ordinary if the derivatives in it are with respect to single independent variable.
- Examples: Let y be the dependent variable and x be the independent variable.

(1)
$$\frac{dy}{dx} + 2y = 6x$$

(2)
$$\frac{d^2y}{dx^2} + 16y = 2x$$

above D.E is O.D.E

Partial Differential Equations:

- Partial Differential equations arise in modeling numerous phenomena in science and engineering.
- A differential equations involves two or more independent variables is called partial differential equation.
- One of the classical partial differential equation of mathematical physics is the equation describing the conduction of heat in a solid body (Originated in the 18th century)
- Examples: Let u is dependent variables and t and x are independent variable.

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

Partial Differential Equation:

Examples:

Laplace Equation

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

u is dependent variable and ***x* , *y*** are independent variables, and is partial differential equation.

2. $$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0$$

3. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

u is dependent variable and ***x* , *t*** are independent variables, and is partial differential equation.

Order of Differential Equation:

- **The order of the differential equation is order of the highest derivative in the differential equation.**

Differential Equation	**ORDER**
$\frac{dy}{dx} = 2x + 3$	1
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$	2
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	3

Degree of Differential Equation:

➢ **The degree of a differential equation is power of the highest order derivative term in the differential equation.**

Differential Equation	**Degree**
$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+ay=0$	1
$\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^4+6y=3$	1
$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^5+3=0$	3

Linear and Non-Linear Differential Equation:

A differential equation is linear, if

1. **dependent variable and its derivatives are of degree one,**
2. **coefficients of a term does not depend upon dependent variable.**

Otherwise is called Non- linear.

Example: 1. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0.$$

is linear.

Example: 2. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

is non - linear because in 2nd term is not of degree one.

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Example: 3. $$x^2 \frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^3$$

is non - linear because in 2nd term coefficient depends on *y*.

Example: 4. $$\frac{dy}{dx} = \sin y$$

is non - linear because $\sin y = y - \frac{y^3}{3!} + -$ is non – linear

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

nth – order linear differential equation

1. nth – order linear differential equation with constant coefficients.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots. + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

2. nth – order linear differential equation with variable coefficients

$$a_n(x)\frac{dy}{dx} + a_{n-1}(x)\frac{d^{n-1} y}{dx^n} + \ldots\ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Solution of Differential Equation

Separable Differential Equations

A separable differential equation can be expressed as the product of a function of x and a function of y.

$$\frac{dy}{dx} = g(x) \cdot h(y) \qquad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2}dy = 2x\ dx$$

$$\int y^{-2}dy = \int 2x\ dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

Combined constants of integration

$$-\frac{1}{x^2 + C} = y$$

$$y = -\frac{1}{x^2 + C}$$

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

→

Families of Solutions:

- General Solution
- Particular Solution
- Singular Solution

➢ A function $y = f(x)$ is called a solution of a differential equation on $-\infty < x < \infty$ or on a finite interval $a < x < b$, if $f(x)$ is continuous and differentiable throughout the interval and if substitution of y(x) and its derivative in to differential equation reduces it to an identity. The general solution of a differential equation of order n contains n arbitrary constants.

➢ General solution is also called the complete integral or complete primitive of the differential equations.

Examples

y=3x+c , is solution of the 1st order differential equation $\frac{dy}{dx} = 3$, c is arbitrary constant. As is solution of the differential equation for every value of c, hence it is known as General solution.

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

- Observe that the set of solutions to the above 1st order equation has one parameter, while the solutions to the above 2nd order equation depends on two parameters.

Particular solutions:

- Any solution obtained from the general solutions, by giving particular values to the arbitrary constants is called a particular solution.
- Examples: consider a first order D.E, $\frac{dy}{dx} = 2xy$

  Hence $y = ce^{x^2}$
- is a general solution but C is arbitrary constant if consider C = 2 or 3 then,

  $y = 2e^{x^2}$ is a Particular solution

  $y = 3e^{x^2}$

- A solution of a differential equation that free from arbitrary parameter is called particular solution.

- <u>Singular Solution</u>: A differential equation possesses a solution that cannot be obtained by specializing any of the parameter in a family of solutions such a solution is called a singular solution.
- However , in some cases there exists a solution which cannot be obtained from the general solutions.
- Usually, singular solutions are of interest only under special topics.
- A non-linear differential equation often has a solution that cannot be obtained from the general solution.
- Consider , $8\frac{dy}{dx}+y^3=0$
- Hence, $y=\frac{2}{\sqrt{x+c}}$ is a General solution but $y=0$ is the Singular solution.

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Initial conditions:

- In many physical problems we need to find the particular solution that satisfies a condition of the **form $y(x_0)=y_0$**. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.
- *Example (cont.):* Find a solution to $y^2 = x^2 + C$ satisfying the initial condition $y(0) = 2$.

$$2^2 = 0^2 + C$$

$$C = 4$$

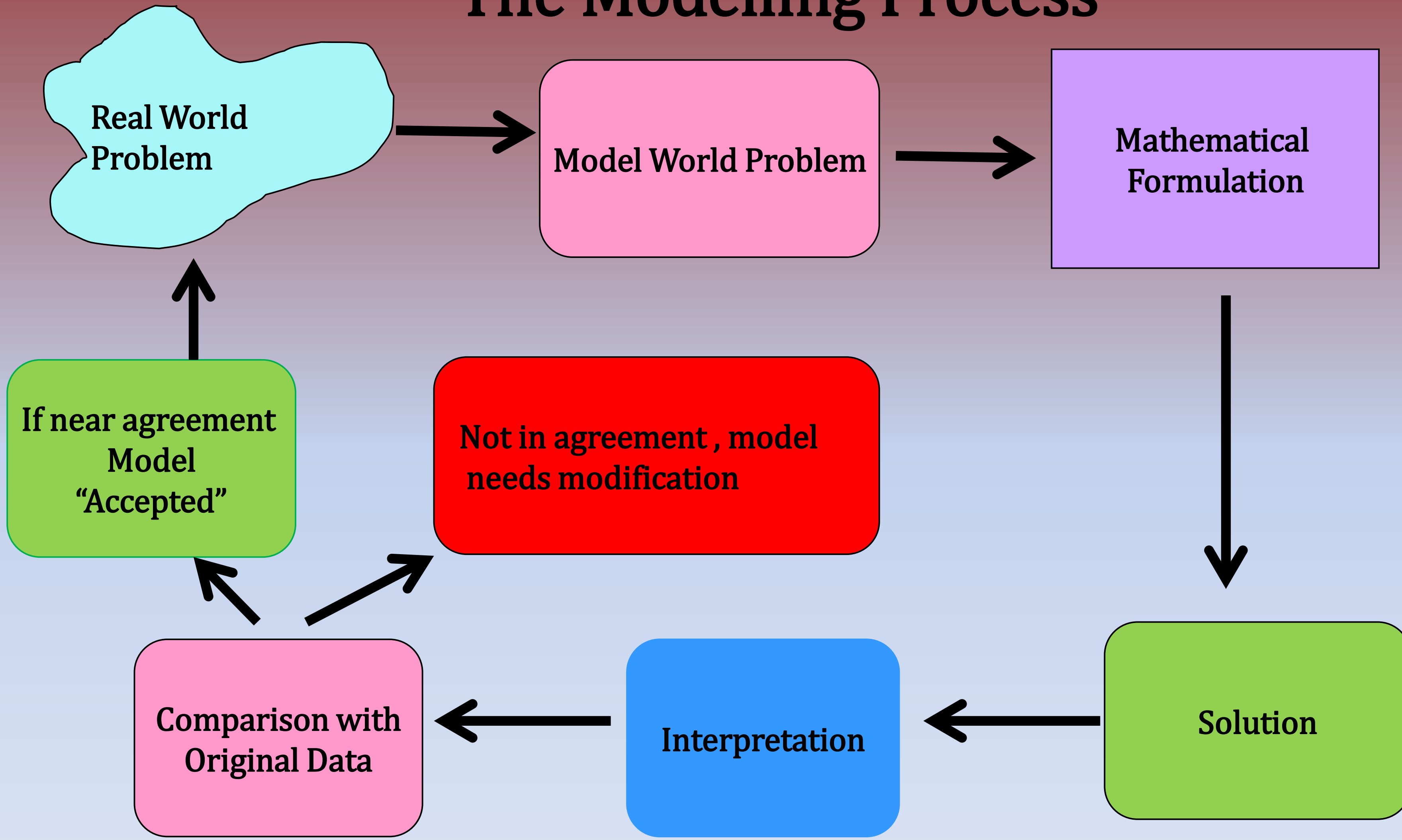
$$y^2 = x^2 + 4$$

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Meaning of Mathematical Modelling

- Real Situations
- Process of Modeling
- **Mathematical Analysis**

The Modelling Process



Mathematical Models

- Our brief discussion process of mathematical modeling
- Idealisation and approximations based on experience and understanding of the real–world situation.
-
- The formulation of a real-world problem in mathematical terms, that is, the construction of a mathematical model.
-
- To make the solution of the resulting mathematical model.
-
- Mathematical analysis based on mathematical experience.
-
- The interpretation of the mathematical results in the context of the original real-world situation by Graphs.

Example of Mathematical Model

- Birth and deaths in a Populations.
- The decay process of radioactive elements.
- Pollution into and out of a lake or river, or the atmosphere.
- Drug assimilation into, and removal from , the bloodstream.

Linear differential equation:

A differential equation is called linear , if

(1) Dependent variable and its derivative are degree one

(2) Coefficients of a term does not dependent variable

A first order differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = q(x) \qquad (1)$$

Where P(x) and q(x) are constants or functions of x alone. Then equation (1) is called linear differential equation in y.

$$\frac{dx}{dy} + p(y)x = q(y) \qquad ....(2)$$

Where p(y) and q(y) are constants or function of y alone. Then eq(2) is linear differential equation in x.

Examples of Linear Differential Equations:

1. $\frac{dy}{dx} + 2y\tan x = \sin x$

2. $(\mathrm{x} + 3\mathrm{y} + 2)(\frac{dy}{dx}) = 1$

3. $\frac{dy}{dx} + y\frac{d\phi}{dx} = \phi(x)\frac{d\phi}{dx}$, where ϕ is a some function of x

4. $\frac{dx}{dy} + yx = e^{y}$

Mr. Ravi Kumar, Assistant
Professor,Department of Mathematics,
Patna Women's College, Patna University.

Way of solution linear differential equation

- First find Integrating Factor (I.F)

$$I.F = e^{\int p(x)dx}$$

The General form of the solution is

$$y.(I.F) = \int q(x).(I.F)dx + \mathrm{c}$$

where c is the arbitrary constants.

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Example:1

Solve the linear differential equation

$$x\frac{dy}{dx}+y+4=0 \qquad \rightarrow 1$$

Solution: Rewriting Eqn. 1.in standard from

Step.1.

$$\frac{dy}{dx}+\frac{y}{x}=\frac{-4}{x} \qquad \rightarrow 2.$$

Here $P(x)=\frac{1}{x}, Q(x)=\frac{-4}{x}$

Step.2. Integrating Factor $I.F=e^{\int P(x)dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

Step. 3. Multiplying Eqn.2. with I.F

$$x\frac{dy}{dx}+\frac{y}{x}.x=-4$$

$$or \quad \frac{d}{dx}(yx)=-4$$

Step. 4. Integrating

$$xy=-4x+c$$

$$or \quad y(x)=-4+\frac{c}{x}$$

is the general solution.

BERNOULLI`S EQUATION

The general form of Bernoulli's equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = Q(x)y^n$$

where n not equal to 0,1 and P and Q are function of x or constant.

Dividing this equation by y^n it can be easily reduced to linear differential equation.

BERNOULL`S EQUATION

Integrating Factor

$$= e^{\int (1-n)p(x)\,dx}$$

Solution form of Bernoulli's equation

$$y^{1-n}\mathbf{e}^{\int (1-n)\,\mathbf{p(x)}\ \mathbf{d}x} = \int (1-n)Q\mathbf{e}^{\int (1-n)^{p}(x)\,\mathbf{d}x}\,\mathbf{d}x + c$$

SOLUTION OF BERNOULLI`S EQUATION

Given equation

$\frac{dy}{dx} + \frac{1}{x}$y= xy^2 … … … … . . (1)

Which is a **Bernoulli's** differential equation.

Here,$p(x) = \frac{1}{x}, Q(x) = X$

And n=2

Integrating factor =$e^{\int(1-n)p(x)dx}$

=$e^{\int p(x)\frac{1}{x}dx}$

=$e^{-\int\frac{1}{x}dx}$

=e^{-lnx}

=$e^{lnx^{-1}}$

=$\frac{1}{x}$

Mr. Ravi Kumar, Assistant
Professor,Department of Mathematics,
Patna Women's College, Patna University.

Solution of (1)will be..

$$y^{1-n}e^{\int(1-n)p(x)dx} = \int (1-n)\, Q\, e^{\int(1-n)\, p(x)\, dx}\, dx + c$$

or, $y^{1-2}\ \frac{1}{x} = \int(1-2)\, x\frac{1}{x}\mathrm{dx} + \mathrm{c}$

or $\frac{1}{y}\frac{1}{x} = -\int dx + c$

or $\frac{1}{xy} = -\int dx + c$

or $\frac{1}{xy} = -\mathrm{x} + \mathrm{c}$

(Ans)

Mr. Ravi Kumar, Assistant
Professor,Department of Mathematics,
Patna Women's College, Patna University.

Exact Differential Equations

- If M and N are function of x and y the equation Mdx+Ndy=0

is called exact when their exist a function f(x, y) of x and y such that

$$d\left[f(x,y)\right]=Mdx+Ndy$$

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=Mdx+Ndy$$

$$\therefore M=\frac{\partial f}{\partial x} \text{ and } \mathrm{N}=\frac{\partial f}{\partial y}$$

Theorem: To determine the necessary and sufficient condition for a differential equation of first order and first degree to be exact.

Working Rule for solving an exact differential equation

1. compare the given equation with $Mdx + Ndy = 0$ and find out M and N

2. Find $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ *if* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

we conclude that the given equation is exact

3. Equate the sum of these two integrals to an arbitrary constant and thus we obtain required solution

$$\underset{\text{treating y as constant}}{\int Mdx} + \int (\text{term in N not containing x})dy = c$$

Example

$(3x^2+4xy)dx+(2x^2+2y)dy=0$ is exact differential equation ,

because it is Total differential of $f(x,y)=x^3+2x^2y+y^2$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x}=3x^2+4xy=M(x,y) \qquad \to .3$$

$$\frac{\partial f}{\partial y}=2x^2+2y=N(x,y) \qquad \to .4$$

$$\begin{aligned} df &= (3x^2+4xy)dx+(2x^2+2y)dy \\ &= M(x,y)dx+N(x,y)dy \end{aligned}$$

Note.1 Eqn. 1 can be written as

$df=0$

by integrating , we obtain the general solution of Eq.1. $f(x,y)=c.$

Note.2. From Eqn. 3 and Eqn.4

$$\frac{\partial^2 f}{\partial y.\partial x}=\frac{\partial M}{\partial y}=4x, \qquad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial x}=4x$$

Mr. Ravi Kumar, Assistant
Professor,Department of Mathematics,
Patna Women's College, Patna University.

Example:5. Show that the differential equation is exact and solve the differential equation.

$$\left(y - xy^2 + 2ye^x\right)dx + \left(x - x^2y + 2e^x\right)dy = 0.$$

Solution: $M(x,y)dx + N(x,y)dy = 0$

$$M(x,y) = y - xy^2 + 2ye^x, \qquad N(x,y) = x - x^2y + 2e^x$$

$$\frac{\partial M}{\partial y} = 1 - 2xy + 2e^x, \qquad \frac{\partial N}{\partial x} = 1 - 2xy + 2e^x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is exact}$$

$$\frac{\partial f}{\partial x} = M(x,y) = y - xy^2 + 2ye^x \qquad \to 1$$

$$\frac{\partial f}{\partial y} = N(x,y) = x - x^2y + 2e^x \qquad \to 2$$

Integrating Eqn.1. w.r.t. x

$$f(x,y) = xy - \frac{x^2y}{2} + 2ye^x + h(y) \quad \to 3$$

Mr. Ravi Kumar, Assistant Professor,Department of Mathematics, Patna Women's College, Patna University.

Differentiation Eqn.3.w.r.t. **y** and equating it to Eqn.2

$$\frac{\partial f}{\partial y} = x - \frac{1}{2}x^2 + 2e^x + \frac{dh}{dy} = x - x^2 y + 2e^x$$

$$\frac{dh}{dy} = -x^2 y + \frac{1}{2}x^2$$

$$h(y) = -\frac{1}{2}x^2 y^2 + \frac{1}{2}x^2 y + c_1$$

Substituting in Eqn.3

$$f(x,y) = xy - \frac{x^2 y^2}{2} + 2ye^x - \frac{1}{2}x^2 y^2 + \frac{1}{2}x^2 y + c_1$$

$$but \;\; f(x,y) = c_2$$

$$xy + 2ye^x - x^2 y^2 + \frac{1}{2}x^2 y + c_1 = c_2$$

$$xy + 2ye^x - x^2 y^2 + \frac{1}{2}x^2 y = c_2 - c_1$$

$xy + 2ye^x - x^2 y^2 + \frac{1}{2}x^2 y = c$ is general solution of the differential equation if $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- **Integrating Factor**: If an equation of the form Mdx+Ndy=0 is not exact , it can always be made exact by multiplying by some function of x and y. Such a multiplier is called an integrating factor.
- **Theorem**: The differential equation Mdx+Ndy =0 possess an infinite number of integrating factors.

Remarks:

- Although an equation of the form Mdx+Ndy =0 always has integrating factors there is no general method of finding them. We now explain rules for finding integrating factors.

Rules for finding Integrating factors

- (1) **By Inspection**: Given equation Mdx+Ndy=0 can be found out by inspection as by rearranging the term of the given equation and/or by dividing by a suitable function of x and y, the equation thus obtained will contain several parts integrable easily.
- (2) If the given equation Mdx+Ndy=0 is homogeneous and

$$(Mx + Ny) \neq 0 \text{ then } \frac{1}{(Mx + Ny)} \text{ is an integrating factor}$$

3. If the equation Mdx+Ndy=0 is off the form

$$f_1(xy)ydx + f_2(xy)xdy = 0 \text{ and}$$

$$(\mathrm{Mx - Ny}) \neq 0 \text{ then } \frac{1}{(\mathrm{Mx - Ny})} \, is \text{ an integrating factor}$$

$$4. \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ is a function x alone}$$

say f(x) then $e^{\int f(x)dx}$ is an integrating factor

of $\mathrm{Mdx + Ndy = 0}$

4. $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function y alone

say f(y) then $e^{\int f(y)dy}$ is an integrating factor

of $Mdx + Ndy = 0$

5. If the given function $Mdx + Ndy = 0$, is of the form

$x^{\alpha} y^{\beta}(mydx + nxdy) = 0$, then

its integrating factor is

$x^{km-1-\alpha} y^{kn-1-\beta}$, where k can have any value.

Refrences

- **M.D. Raisinghania, Ordinary and Partial Differential equations , Revised Edition(2011)**
- **B. Barnes and G.R. Fulford, Mathematical Modeling With Case Studies Using Maple and MATLAB, Third Edition, CRC Press**

(Special Indian Edition) (2015)